Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

The Elliptic Inequalities in the Lunar Theory.

By Ernest W. Brown, *Fellow of Christ's College, Cambridge.*

The object of this paper is to give a general solution in series of the problem "A system of three bodies is in motion in one plane, the first is revolving about the second and is disturbed from its elliptic orbit by the third. The third body is supposed to be of infinite mass and to be moving in a circle of infinite radius with a finite angular velocity. Given the relative positions of the three bodies at any one time, to find their relative positions at any other time." The differential equations which express the motion of the first body relative to the second have been given by Dr. Hill in Vol. I of this journal. The second body is taken as the origin, and the motion is referred to rectangular axes moving with the angular velocity of the third body which always lies on the x-axis. The equations are

$$\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} + \left(\frac{\mu}{r^3} - 3n'^2\right)x = 0,$$

$$\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} + \frac{\mu}{r^3}y = 0,$$

where μ is the sum of the masses of the first and second bodies and n' the angular velocity of the third about the origin.

One integral of these equations is immediately obtained. It is

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{2\mu}{r} + 3n'^2x^2 - 2C.$$

In chapter II of his memoir just referred to, Dr. Hill transforms these equations to the imaginary variables $x \pm y\sqrt{-1}$, and by the use of this, the Jacobian Integral, brings them to two simultaneous differential equations of the second order and second degree.

A particular solution is then found which can be expressed by an infinite trigonometrical series with a single period, giving as an orbit relative to the moving axes, a closed symmetrical curve which, for small values of the ratio of the mean motions of the first and third body, is an oval. I have given the extension of this particular solution to the case where the third body is at a comparatively great but not infinite distance.* The orbit is still a closed curve, but it has lost its symmetry about the axis of x. This extension presented no special difficulties beyond the question of representing the series obtained in the most convergent form.

When an attempt is made to obtain the general solution, it appears below that it will be necessary to represent it by an infinite series with two periods; that is, by a doubly infinite trigonometrical series. The orbit is now no longer a closed curve, but it is one which re-enters its orbit after an infinite time, and the maxima and minima values of the coordinates lie between determinate limits. The chief difficulty lies in the fact that the equations of condition between the coefficients of the terms in the series require a relation between the two periods; it is the finding of this relation that entails the trouble.

Again, since the solution is expressed by an infinite series, it is necessary in order that it should have a meaning, that the series should be convergent, for this, two of the arbitrary constants must be confined within certain limits. We are thereby limited to consider only disturbing bodies whose mean motions have ratios to that of the revolving body which do not surpass a certain value, and also that the body when undisturbed shall have an eccentricity less than some given quantity. It does not seem easy to determine what these limits are or whether they exist at all. M. Poincaré's remarkable researches† into the methods of the problem of three bodies have to a large extent opened the way to the determination of convergency. He has in fact proved that for small values of certain of the constants, there are periodic solutions which can be represented by convergent series. The convergence of the series obtained below will depend in a large measure on the value of c, the ratio of the two periods of the general solution.

The principal part of the ratio c has been shown to be one of the roots of an equation expressed as an infinite determinant, and it was this peculiarity that stimulated inquiry into the possibility of the existence of such forms. G. W. Hill, Poincaré and Helge von Koch have given investigations of their properties. In particular, Dr. Hill,* under the assumption that it was possible, actually obtained the principal part of c from the infinite determinant which arose in his method; later M. Poincaré† gave conditions which, when satisfied, allowed the determinant to possess a finite and determinate value, and also proved that it was possible to obtain c from it; and lastly, Dr. von Koch‡ has shown under what circumstances the ordinary transformations used in determinants with a finite number of rows and columns can be extended when these latter are infinite in number.

---

* American Journal of Mathematics, Vol. XIV, p. 141.

† Acta Mathematica, Vol. XIII. Mécanique Celeste, Vol. I.

If we examine the series given by Delaunay in longitude and parallax for the coefficients of the several trigonometrical terms, it appears that at any rate in the case of the moon, slow convergence, whenever it occurs, proceeds in any coefficient along the series arranged in powers of the ratio of the mean motions; also that in terms with actually small coefficients the convergence appears to be slower than in the larger coefficients. As pointed out by Dr. Hill, it becomes of interest to see how these can be improved either by taking new quantities in terms of which to make the expansions or by using numerical values from the start. He has examined this question in the case of the principal part of the Variational Inequalities, and I have attempted to give convenient expressions for the same purpose in that of the Parallactic Inequalities. When the equations for the Elliptic Inequalities are examined, it does not appear that there is any method of improving the expressions for the coefficients on the same lines beyond taking $m = n'/(n - n')$ instead of n'/n in terms of which to make expansion. The quantity c, which is itself represented by a slowly converging series in powers of m, is involved in a complex manner, and even if it were not, we should simply be brought back to the difficulty of representing c by a quickly convergent series. Hence it seems better to use the numerical value of m from the outset, *keeping all the other constants involved arbitrary*; that is to say, in the case treated here the eccentricity is left arbitrary.

Another consideration justifies this method of proceeding. Of all the constants used in the determination of the motion of the moon, the ratio of the mean motions is most exactly determined, and it is doubtful whether any future observations will change it to any appreciable extent. This is not the case with the other constants. As the slow convergence occurs in the series arranged in powers of this ratio, little will be lost by keeping the rest of the constants arbitrary, and the extra trouble on the plan here adopted is very small. If then a complete theory of the moon were given on this principle, the coefficient of every term would appear as a quickly convergent series arranged in powers of the eccentricities, the ratio of the parallaxes, and the sine of the latitude, with numerical multipliers only. The coefficients in such a development would have a comparatively small number of terms with a high degree of accuracy.

---

* Acta Math., Vol. VIII.

† Bulletin de la Soc. Math. de France, Vol. XIV, p. 77.

‡ Acta Math., Vol. XVI, p. 217.

Another point may also be noted in connection with these methods It appears generally that in proceeding to find the coefficients by successive approximation, the second and higher approximations are due principally, not to the introduction of new terms in the series, but to the second and higher approximations of the terms previously used, so that it is not generally necessary to find the first approximations of very many of the higher terms in order to get the large coefficients accurately. This fact abbreviates the calculations to a large extent.

With respect to the numerical results obtained, no trouble has been spared to make them accurate. Equations of verification have been computed for each step and also for the final results. The various forms which the equations of motion can take enable this to be easily done. In the second portion of the paper (to be published hereafter) a value has been obtained for the part of the motion of the Lunar Perigee which depends on the square of the eccentricity. In order that this value should possess a degree of accuracy which can be relied on, it is absolutely necessary that the quantities obtained be verified by independent equations. I have not had the advantage of any assistance in carrying out the computations, and have consequently taken special pains to find out the best equations for verification, and have computed in certain cases two and even three of them to test the same results.

I.

The equations under discussion are

$$\frac{d^2x}{dt^2} - 2n' \frac{dy}{dt} + \left(\frac{\mu}{r^3} - 3n'^2\right) x = 0,$$

$$\frac{d^2y}{dt^2} + 2n' \frac{dx}{dt} + \frac{\mu}{r^3} \cdot y \qquad = 0,$$

which admit of a general solution when n' is zero and a particular solution when n' is not zero. In the first case the resulting orbit is an ellipse, and therefore, as is well known, x, y can be expressed in terms of the time by means of Bessel's functions*

$$x = a \sum_{-\infty}^{+\infty} {}_p \frac{1}{p} J_{\frac{pe}{2}}^{(p-1)} \cos pg\,(t - t_0),$$

$$y = b \sum_{-\infty}^{+\infty} {}_p \frac{1}{p} J_{\frac{pe}{2}}^{(p-1)} \sin pg\,(t - t_0),$$

where $\frac{1}{0} J_0^{(-1)} = -\frac{3}{2} e$, $b^2 = a^2 (1 - e^2)$ and g is the mean motion. The four quantities, a, e, g, t_0, involve three arbitrary constants, the fourth being used in making the major axis of the ellipse coincide with the x-axis. The particular solution of the equations involves only two arbitrary constants, and is of the form†

$$x = a_0 \sum_{-\infty}^{+\infty} {}_i\, a_i \cos (2i + 1)(n - n')(t - t_1),$$

$$y = a_0 \sum_{-\infty}^{+\infty} {}_i\, a_i \sin (2i + 1)(n - n')(t - t_1),$$

the arbitraries being n (or a_0) and t_1 (when $i = 0$, $a_i = 1$). These two forms are distinct unless both e and n' are zero, when they reduce to circular motion. The coefficients a_i are functions of the ratio $n' : n - n'$. Since then we have these forms for zero values of n' and e respectively, *it is assumed* that for small values of e and n' there is a possible solution of the form

$$\left.\begin{aligned} x &= a_0 \Sigma_i \Sigma_p A_{i,\,p} \cos \{(2i + 1)(n - n')(t - t_1) + pg\,(t - t_0)\}, \\ y &= a_0 \Sigma_i \Sigma_p A_{i,\,p} \sin \{(2i + 1)(n - n')(t - t_1) + pg\,(t - t_0)\}, \end{aligned}\right. \quad (i, p = +\infty \ldots\ldots -\infty).$$

This solution involves the necessary four arbitrary constants and may be looked upon as a general solution of the equations except in so far as e and $n'/(n - n')$ are limited in magnitude. This limitation is necessary in order that the series may be convergent, but it is to be understood that there is no guarantee that the series are convergent for even small values of these constants. At present all we know is that the solution with n' zero converges for values of e sufficiently small, and the particular solution for sufficiently small values of the ratio $n'/(n-n')$.

---

* Tisserand, Méc. Cel. t. I, p. 226.

† G. W. Hill, Researches in the Lunar Theory, Amer. Jour. Math., Vol. I, p. 130. Unless otherwise stated, the references to this memoir are given by the page of Vol. I of this journal.

The relation $g:n-n'$ will involve some of the constants. Put then

$$g = c(n-n'),$$
$$\alpha = \{(2i+1)(t-t_1) + pc(t-t_0)\}.$$

Transform the equations of motion by putting $u = x + y\sqrt{-1}$, $s = x - y\sqrt{-1}$, and

$$-\frac{\sqrt{-1}}{n-n'}\cdot\frac{d}{dt} = D, \quad \frac{n'}{n-n'} = m;$$

they become, with the help of the Jacobian Integral (Vol. I, p. 132),

$$\left.\begin{aligned} D^2(us) - Du.Ds - 2m(uDs - sDu) + \frac{9}{4}m^2(u+s)^2 &= C, \\ D(uDs - sDu - 2mus) + \frac{3}{2}m^2(u^2 - s^2) \qquad &= 0. \end{aligned}\right\} \qquad (1)$$

We have
$$D(\alpha\sqrt{-1}) = 2i + 1 + cp,$$

and since the coefficients of the various powers of $e^{\alpha\sqrt{-1}}$ for all values of i, p (except the continuation $i = p = 0$) will be equated to zero, the constants t_0 and t_1 will not enter into the equations of condition thus found. They may be therefore omitted in what follows, care being taken that they be inserted when the values of the coefficients and of c have been found and the series written out in full.

Hence we can put
$$\alpha = (2i + 1 + cp)(n-n')\tau,$$

where $d\tau = dt$. Also write
$$e^{(n-n')\tau\sqrt{-1}} = \zeta,$$

so that
$$D = \zeta\frac{d}{d\zeta};$$

and the assumed solutions become

$$\left.\begin{aligned} u &= \Sigma_i\Sigma_p A_{i,\,p}\,\zeta^{2i+1+cp}, \\ s &= \Sigma_i\Sigma_p A_{-i-1,\,-p}\,\zeta^{2i+1+cp}, \end{aligned}\right\} \qquad (2)$$

Substitute these in the equations (1). For this purpose

$$
\begin{aligned}
Du &= a_0\Sigma_i\Sigma_p\,(2i+1+cp)\,A_{i,\,p}\,\zeta^{2i+1+cp},\\
Ds &= a_0\Sigma_i\Sigma_p\,(2i+1+cp)\,A_{-i-1,\,-p}\,\zeta^{2i+1+cp},\\
us &= a_0^2\Sigma_i\Sigma_p\Sigma_j\Sigma_q A_{j,\,q}A_{-i+j,\,-p+q}\,\zeta^{2i+cp},\\
&\text{etc.,}
\end{aligned}
$$

where i, j, p, q take all integral values from $+\infty$ to $-\infty$. Equate to zero the coefficients of the various powers of ζ for all values of i, p, in order to obtain the requisite equations of condition between the coefficients. It is necessary that cp should not be an integer for any value of p; otherwise a set of equations of condition would be lost and an indeterminateness would result. As far as can be seen, cp is not an integer for any finite values of p. With this proviso there results

$$
\left.
\begin{aligned}
&\Sigma_j\Sigma_q[(2i+cp)^2-(2j+1+cq)(2i+cp-2j-cq-1)-2m(2i+cp-4j-2cq-2)\\
&\qquad+\tfrac{9}{2}m^2]\,A_{j,\,q}A_{j-i,\,q-p}+\tfrac{9}{4}m^2\Sigma_j\Sigma_q\,(A_{j,\,q}A_{i-j-1,\,p-q}+A_{j,\,q}A_{-i-j-1,\,-p-q})=0,\\
&(2i+cp)\,\Sigma_j\Sigma_q[2i+cp-4j-2cq-2-2m]\,A_{j,\,q}A_{j-i,\,q-p}\\
&\qquad\qquad+\tfrac{3}{2}m^2\Sigma_j\Sigma_q(A_{j,\,q}A_{i-j-1,\,p-q}-A_{j,\,q}A_{-i-j-1,\,-p-q})=0,
\end{aligned}
\right\}\ (3)
$$

where in these equations j and q take all values from $+\infty$ to $-\infty$, and the equations are true for all values of i and p within the same range with a single exception—when $i=p=0$, the right-hand side of the first equation is C, the constant of the Jacobian integral.

It will be noticed, either from the way in which these equations are obtained or immediately from their form, that the coefficients in square brackets are directly deducible from those of Hill's equations (Vol. I, p. 133) by putting in the latter $i+cp$ for i and $j+cq$ for j. The process he gives for isolating the terms of principal importance in corresponding pairs of equations can therefore be employed here.

Multiply the first equation by 2 and subtract the second multiplied by 3; we obtain

$$\Sigma_j\Sigma_q\,[f(j,\,i,\,p,\,q)\,A_{j-i,\,q-p}A_{j,\,q}+9m^2A_{-i-j-1,\,-p-q}A_{j,\,q}]=0\,,$$

where

$$
\begin{aligned}
f(j,\,i,\,p,\,q) = -\,(2i+cp)^2+4\,(2i+cp)+2+4\,(2i+cp+1)(2j+cq)&\\
+\,2\,(2j+cq)^2+2m\,\{2i+cp+4\,(2j+cq)+4\}+9m^2.&
\end{aligned}
$$

This single equation includes both equations by giving positive and negative values to i and p. The previous equations were sufficient if only positive *or* negative values were taken. A corresponding pair of equations is defined by the values i and p, $-i$ and $-p$. The terms of principal importance for obtaining $A_{i,\,p}$ and $A_{-i,\,-p}$ are got by putting in the first of this pair of equations $q=p, j=i$ and $q=0, j=0$, and in the second $q=0$, $j=0$ and $q=-p$, $j=-i$. These terms are respectively then

$$\begin{aligned}&f(i,\ i,\ p,\ p)\,A_{i,\,p}A_{0,\,0}+f(0,\ i,\ p,\ 0)\,A_{-i,\,-p}A_{0,\,0},\\&f(0,\ -i,\ -p,\ 0)\,A_{i,\,p}A_{0,\,0}+f(-i,\ -i,\ -p,\ -p)\,A_{-i,\,-p}A_{0,\,0}.\end{aligned}$$

Regard the equations as simultaneous ones to determine $A_{i,\,p}$ and $A_{-i,\,-p}$, solve so as to obtain $A_{i,\,p}$ and $A_{-i,\,-p}$ separately. The common divisor is

$$12\,(2i+cp)^2\,[2\,(2i+cp+1)(2i+cp-1)-4m+m^2],$$

and either equation is included in

$$\left.\begin{aligned}\Sigma_j\Sigma_q\,\{(j,\,i,\,p,\,q)\,A_{j,\,q}A_{j-i,\,q-p}+(i,\,p)\,A_{j,\,q}A_{i-j-1,\,p-q}&\\+[i,\,p]\,A_{j,\,q}A_{-i-j-1,\,-p-q}\}=0,&\end{aligned}\right\}\quad(4)$$

where

$$(j,\,i,\,p,\,q)=-\frac{\begin{aligned}(2j+cq)\{(2i+cp)^2-(2+4m-m^2)+(2i+cp)(2+2m)\}&\\+(2j+cq)^2(2i+cp-2-2m)&\end{aligned}}{(2i+cp)\{2\,(2i+cp)^2-2-4m+m^2\}},$$

$$(i,\,p)=-\frac{3m^2}{4\,(2i+cp)^2}\,\frac{(2i+cp-2)(2i+cp-2-2m)-6-12m-9m^2}{2\,(2i+cp)^2-2-4m+m^2},$$

$$[i,\,p]=-\frac{3m^2}{4\,(2i+cp)^2}\,\frac{(2+10i+5cp)(2i+cp-2-2m)+6+12m+9m^2}{2\,(2i+cp)^2-2-4m+m^2}.$$

In this equation the multiplier of $A_{i,\,p}A_{0,\,0}$ or $(i,\,i,\,p,\,p)$ is -1 and that of $A_{-i,\,-p}\,A_{0,\,0}$ or $(0,\,i,\,p,\,0)$ is 0. By making the substitutions mentioned above, this equation might also have been deduced directly (Vol. I, p. 135). It is evident that all through the transformations cp is associated with $2i$ and cq with $2j$. The single equation (4) includes all the equations when i and p receive positive and negative values. When $i=p=0$, the right-hand side is no longer zero, but depends on the constant of Jacobi's integral and connects the latter with the other four constants introduced into the assumed solution.

Let us look for a moment at the way in which the primary forms are obtained from this general form. For the first form take the set of equations given by $i = 0$ and p all integral values from $+\infty$ to $-\infty$, and put in them $m = 0$; since they reduce to the given forms for the first solution, this entails $j = 0$ only. In addition as $n' = 0$ and $g = cn$, n becomes equal to g and $c = 1$. The coefficients (i, p) and $[i, p]$ disappear, and we have for the determination of the Bessel's functions

$$\Sigma_q \frac{q}{p}\left[1 - \frac{(p-q)(p-2)}{2p^2 - 2}\right] A_{0,\,q} A_{0,\,q-p} = 0, \quad p = +\infty \ldots\ldots -\infty$$

or $$\Sigma_q q\left[p^2 + q(p-2) + 2p - 2\right] A_{0,\,q} A_{0,\,q-p} = 0,$$

(except when $p = 0$, when the right-hand side is C), and the assumed solution becomes, when the axis of the ellipse is made to coincide with the axis of x; that is, when $t_1 = t_0$,

$$x = a_0 \Sigma_p A_{0,\,p} \cos(p+1) g(t - t_0),$$
$$y = a_0 \Sigma_p A_{0,\,p} \sin(p+1) g(t - t_0).$$

Comparing, we obtain

$$a_0 A_{0,\,0} + a_0 A_{0,\,-2} = a,$$
$$a_0 A_{0,\,0} - a_0 A_{0,\,-2} = b,$$

and as a first approximation

$$A_{0,\,0} = 1, \quad A_{0,\,-1} = -\tfrac{3}{2} e, \quad A_{0,\,1} = e,$$

and $A_{0,\,p}$ is of the order e^p at least.

The particular solution is deduced by making $p = 0$ and $q = 0$ or $e = 0$, when it reduces to that given by Dr. Hill. Hence $a_0 . A_{i,\,0} = \mathrm{a}_i$, in his notation. As a_i is of the order m^{2i} at least, it is evident that unless the denominators contain powers of m as factors, $A_{i,\,p}$ is of the order $m^{2i} e^p$ at least.

The equations can then, provided m and e be sufficiently small, be solved by successive approximation, and this method is used below. In certain cases a power of m will appear in the denominator, but this does not affect the principle, as even then each approximation carries us at least two orders higher in m; in the solution given above it is impossible for more than the first power of m to be contained as a factor in the denominator.

II.

The system of denominators

$$(2i + cp)\left[2(2i + cp)^2 - 2 - 4m + m^2\right]$$

deserves special examination. The value of c is discussed below. For the purpose in view here, I assume that it is known and that it has the expression in series given by Delaunay* or the numerical value given by Dr. Hill for its principal part.† These are

$$c = 1.071583 \ldots.,$$
$$c = 1 + m - \tfrac{3}{4} m^2 - \tfrac{201}{32} m^3 \ldots., \text{ where } m = n'/(n - n').$$

Consider the algebraical expression of the two factors of this denominator-system when the second value of c has been substituted. The terms independent of m can only vanish in the first factor when $p = -2i$, and in the second factor when $p = -2i \pm 1$. Hence the coefficients $A_{i,\,-2i}$, $A_{i,\,-2i\pm 1}$ will have their orders lowered by at least one power of m due to this cause. Further, they cannot have their orders lowered by more than one power of m from these denominators, since the coefficient of m cannot vanish while that independent of m vanishes for any values except when $i = 0$, $p = \pm 1$, a case which does not come in here. Owing to the fact, however, that any coefficient $A_{i,\,p}$ will depend on those of lower orders, this rule does not give the lowest order. It can be easily seen from the equations (4) what the order of any coefficient is. In particular, we have

$$\left.\begin{array}{ll} p = +1 & i = -1 \\ p = -1 & i = +1 \end{array}\right\} A_{i,\,-1},\ A_{-i,\,1} \text{ of order } em^{2i-1}, \qquad (i = 1,\ 2 \ldots. \infty)$$
$$\left.\begin{array}{ll} p = +2 & i = -1 \\ p = -2 & i = +1 \end{array}\right\} \begin{array}{l} A_{i,\,-2},\ A_{-i,\,2} \text{ of order } e^2 m^{2i-2}, \qquad (i = 2,\ 3 \ldots. \infty) \\ A_{1,\,-2},\ A_{-1,\,2} \text{ of order } e^2 m, \end{array}$$

and so on. These are well known, but the simplicity with which they are obtained seems worth mentioning.

Returning to numerical values, the equations

$$2i + p \times 1.071583 \ldots. = 0,$$
$$2i + p \times 1.071583 \ldots. = \pm\sqrt{1 - 2m + m^2/2}.$$
$$= \pm\, 1.076303 \ldots.$$

solved *approximately* in integers will give those coefficients whose expressions have small denominators. The set of values

$$p = -2i \text{ or } -2i \pm 1, \qquad (i = +\infty \ldots. -\infty)$$

---

* Comptes Rendus, t. LXXIV, p. 19.

† Acta Math., Vol. VIII., p. 36.

will as before give a series of such coefficients, but the denominators increase with the value of i. Again

$$p = \pm 15, \quad i = \mp 8$$

will be the smallest denominator of another series from the first equation and

$$p = \pm 14, \quad i = \mp 8$$

from the second equation. In this way it is possible to find those terms with comparatively large coefficients.

If we had included the terms due to parallax, solar eccentricity and inclination, the denominators would have been of the form

$$(i + cp + gq + rm)[2(i + cp + gq + rm)^2 - 2 - 4m + m^2],$$

where i, p, q, r receive all positive and negative integral values, and where g is the relation of the motion of the node to the difference of the mean motions. The values for which the first factor becomes small are well known, since it occurs directly in integrating the equations used in the older methods. The equation.

$$i + cp + gq + rm = \pm\sqrt{1 - 2m + m^2/2}$$

will have approximate solutions in integers, and those solutions in which i, p, q, r are not very large will determine coefficients great in comparison with their apparent order. These trial solutions would be obtained by inserting the numerical values of c, g, m.

The terms obtained by making the first factor small are the well-known long-period terms; those obtained from the second factor have periods of nearly a month. This closer approximation than the value unity, usually taken instead of the expression $\sqrt{1 - 2m + m^2/2}$, might reveal new terms hitherto neglected. The monthly differences between observation and the theoretical tables are still sufficiently large to justify an examination of these short-period inequalities.

III.

Determination of the Coefficients dependent on the First Power of the Eccentricity.

It has been shown that $A_{i,p}$ is at least of the order e^p in e. In order therefore to obtain the values of the coefficients depending on the first power of e only, p will have the values $+1$ and -1, while the corresponding values which q can take are $+1, 0$ and $0, -1$. Put for convenience

$$A_{i,\,0} = a_i,^* \quad A_{i,\,1} = \varepsilon_i, \quad A_{i,\,-1} = \varepsilon_i',$$

the equations (4) giving ε_i and ε_{-i}' will become

$$\left.\begin{aligned} &\Sigma_j \{ (j,\ i,\ 1,\ 1)\, \varepsilon_j a_{j-i} + (j+i,\ i,\ 1,\ 0)\, \varepsilon_j' a_{j+i} \\ &\qquad\qquad + 2(i,\ 1)\, \varepsilon_j a_{i-j-1} + 2\,[i,\ 1]\, \varepsilon_j' a_{-i-j-1} \} = 0, \\ &\Sigma_j \{ (j-i,\ i,\ -1,\ 0)\, \varepsilon_j a_{j-i} + (j,\ -i,\ -1,\ -1)\, \varepsilon_j' a_{j+i} \\ &\qquad\qquad + 2\,[-i,\ -1]\, \varepsilon_j a_{i-j-1} + 2(-i,\ -1)\, \varepsilon_j' a_{-i-j-1} \} = 0. \end{aligned}\right\} \quad (5)$$

Since the quantities a_i are known, these equations will suffice to determine the relations of ε_i and ε_{-i}' to ε_0 or ε_0'. There are in all as many equations as there are unknowns ε_i, ε_{-i}'. In order therefore that these equations may be consistent, some relation must hold between the quantities a_i and the coefficients; it is from this relation that the value of c is to be found. For the sake of the argument which follows, the equations will be given another form.

In (3) put $p = 1$, $q = 0, 1$, add and subtract the results. We obtain

$$\left.\begin{aligned} &\Sigma_j \{ c^2 a_{j-i} + G_{j,\,i}\, a_{j-i} + G_{j+i,\,i}\, a_{j+i} + \tfrac{3}{4} m^2 (a_{i-j-1} + 5a_{-i-j-1}) \} X_j \\ &\qquad\qquad + c\Sigma_j (H_{j,\,i}\, a_{j-i} - H_j' a_{j+i})\, Y_j = 0, \\ &\Sigma_j \{ c^2 a_{j-i} + G_{j,\,i}\, a_{j-i} - G_{j+i,\,i}\, a_{j+i} + \tfrac{3}{4} m^2 (a_{i-j-1} - 5a_{-i-j-1}) \} Y_j \\ &\qquad\qquad + c\Sigma_j (H_{j,\,i}\, a_{j-i} + H_j' a_{j+i})\, X_j = 0, \end{aligned}\right\} \quad (6)$$

where

$$\begin{aligned} G_{j,\,i} &= \tfrac{1}{2}(2j+1)^2 + (2j+1)(i+2m) + \tfrac{9}{4} m^2, \\ H_{j,\,i} &= \tfrac{3}{2}(2j+1) + i + 2m, \qquad & H_j' &= \tfrac{1}{2}(2j+1), \\ X_j &= \varepsilon_j + \varepsilon_j', & Y_j &= \varepsilon_j - \varepsilon_j'. \end{aligned}$$

Let ∞ denote the number of integers in the sequence $1, 2, 3 \ldots\ldots \infty$, there are then $4 \times \infty + 2$ of these equations and the same number of unknowns X_j and Y_j. As these equations are linear, suppose the unknowns eliminated by a determinant, which may be looked upon as in equation in c having $4 \times \infty + 2$ roots. It is immediately evident from its form that it involves only even powers of c, since the coefficients G, H, H' do not contain c.

The above equations are convergent series when $|c^2 Y_j|$, $|c^2 X_j|$ are not infinite for any value of j, and the series $\Sigma |a_i|$ is absolutely convergent.

---

* The coefficient here called a_i is therefore the same as Dr. Hill's a_i/a_0. The change is convenient, since we are principally concerned with the ratios of the coefficients to a_0; a_i will be a numerical quantity ultimately.

In order to find what values of c will make these equations consistent, it is possible to proceed as follows. In either of equations (3) put $c + 2s$ for c and $i - s$ for i, where s is an integer, the equation considered will remain unaltered. Hence the result of the first substitution causes the equation with suffix i to become that with suffix $i + s$. Any such change therefore merely permutes the equations into one another. Hence if c_0 be any value of c which renders the equations consistent, any one of the set of values

$$\pm c_0, \quad \pm (c_0 \pm 2), \quad \pm (c_0 \pm 4), \quad \pm (c_0 \pm 6), \ldots .$$

will have the same effect. If we form a determinant by eliminating the X_j and Y_j, and if this determinantal equation for c have one root c_0, it will have as roots the set of values given above. This determines $2 \times \infty + 1$ values of c^2 when c_0 is known. As there are $4 \times \infty + 2$ values altogether, there are two values of c^2 which are both non-infinite and which do not differ by any even integer. It may be shown that there is one value of c_0^2 which is zero. Dr. Hill,* in obtaining the other value of c_0^2, submitted equations (1) to a variation δ and obtained two equations linear in δu, δs. A solution of these equations might have been obtained by putting

$$\delta u = \Sigma_i \varepsilon_i \zeta^{2i+1+c} + \Sigma_i e_i' \zeta^{2i+1-c}$$

and the corresponding value for δs, that is the solution considered above when the second power of the eccentricity is neglected. He points out that

$$\delta u = Du, \quad \delta s = Ds,$$

where u, s represent the second particular solution is also a solution of the equations. Now Du, Ds are odd power series in ζ, and therefore the above solution demands that c shall have an even integral value. We thus obtain the result that the roots of the determinant in c^2 are

$$\left.\begin{matrix} c_0^2, \ (c_0 \pm 2)^2, \ (c_0 \pm 4)^2, \ldots . (c_0 \pm 2i)^2, \ldots . \\ 0^2, \ 2^2, \ 2^2, \ 4^2, \ 4^2, \ldots . (2i)^2, \ (2i)^2, \ldots . \end{matrix}\right\} i = 1, 2 \ldots . \infty .$$

The second series of values does not appear in Dr. Hill's determinant. He puts

$$\delta u = v . Du, \quad \delta s = w . Ds$$

---

* On the Part of the Motion of the Lunar Perigee, etc., Acta Math., Vol. VIII.

and forms the equations in v, w; these are reduced so that v and w can both be expressed by a single equation

$$D^2 \mathrm{w} = \Theta \mathrm{w},$$

where Θ is a known function. The determinant which is ultimately formed then consists of an equation in c^2 which has only $2 \times \infty + 1$ roots, comprising those of the first series above.

It might seem that it was now possible to proceed to obtain the value of c_0 as Dr. Hill has done in the memoir just referred to. When this is attempted, however, we are met at the outset by the difficulty that the determinant is not convergent. The coefficient of the highest power of c^2 in it is the square of

$$\left|\begin{array}{ccccc} \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\, a_{-1} & 1 & a_1 & a_2 & a_3 \cdots \\ \cdots\, a_{-2} & a_{-1} & 1 & a_1 & a_2 \cdots \\ \cdots\, a_{-3} & a_{-2} & a_{-1} & 1 & a_1 \cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \end{array}\right| \equiv \nabla,$$

which does not satisfy the second condition given by M. Poincaré, namely, that the sum of the absolute values of the non-diagonal elements should be finite.* Also it does not seem possible to find any set of factors which would render this determinant at once finite and adapted to calculation.

Let $\Delta(c)$ denote the determinant form obtained from equations (6). We have

$$(\cos \pi c - 1)(\cos \pi c - \cos \pi c_0) \equiv \Delta'(c) \div \nabla^2,$$

where $\Delta'(c)$ has been deduced from $\Delta(c)$ by suitably multiplying the rows of this latter so as to make the coefficients of the highest power of c^2 on each side of the identity the same. The determinants $\Delta'(c)$ and ∇ are not finite, and it does not seem possible to find any set of factors to make them both so. All that can be said at present is this: as c_0 is known to have a determinate and finite value (in the case of our moon), the ratio $\Delta'(c)/\nabla^2$ is not infinite for finite values of c and will not be zero unless the value given to c be one of the roots of the equation

$$(\cos \pi c - 1)(\cos \pi c - \cos \pi c_0) = 0.$$

For the calculation of c_0 this method appears to be of no use, but the algebraical forms which it brings forward are not devoid of interest.

---

* Bulletin de la Société mathématique de France, t. XIV, pp. 77–90.

The value of c can be calculated by continued approximation in connection with the coefficients ε_i, ε'_i when we know a first approximation to it. It is easily seen that the value we want is nearly unity; from equations (5) this would give as a real first approximation

$$c = + \sqrt{\left(1 - 2m + \frac{m^2}{2}\right)}.$$

As a calculation by this method would involve recomputing the coefficients in (5) at each step, the method would be exceedingly laborious. I have therefore thought it best, instead of attempting to find c, to assume Dr. Hill's value and calculate the corresponding values of ε_i and ε'_i in terms of ε_0 and ε'_0. There are then unused the two equations for ε_0 and ε'_0, which by substitution of the values of ε_i and ε'_i should give the same ratio $\varepsilon_0/\varepsilon'_0$ if the assumed value of c is correct. Since the equations are all linear, this is not difficult, and such a proceeding will give a real verification of the value of c as far as the approximations are taken.

IV.

Calculation of the Coefficients ε_i and ε'_i.

In order to calculate the coefficients in the equations (5), it is convenient to put them into that form which will render the labor as small as possible. Let

$$\left.\begin{matrix} L_i \\ L'_i \end{matrix}\right\} = -\frac{(2i \pm c)^2 - (2 + 4m - m^2) + (2i \pm c)(2 + 2m)}{(2i \pm c)\{2(2i \pm c)^2 - 2 - 4m + m^2\}},$$

$$\left.\begin{matrix} M_i \\ M'_i \end{matrix}\right\} = -\frac{2i \pm c - 2 - 2m}{(2i \pm c)\{2(2i \pm c)^2 - 2 - 4m + m^2\}},$$

where L_i, M_i refer to the upper signs and L'_i, M'_i to the lower. We then have

$$f(i, j, \quad 1, q) = (2j + qc)\, L_i + (2j + qc)^2 M_i,$$
$$f(i, j, -1, q) = (2j + qc)\, L'_i + (2j + qc)^2 M'_i.$$

When we neglect all terms except those depending on the first power of the eccentricity, q takes the values 1, 0 in the first of these expressions and the values 0, -1 in the second; and the first terms in equations (5) take the forms

$$\Sigma_j \left[P_{j,\,i}\, a_{j-i}\varepsilon_j + Q_{j,\,i}\, a_j \varepsilon'_{j-i}\right],$$
$$\Sigma_j \left[Q'_{j,\,i}\, a_{j-i}\varepsilon_j + P'_{j,\,i}\, a_j \varepsilon'_{j-i}\right],$$

where

$$P_{j,\,i} = (2j + c)\,L_i + (2j + c)^2 M_i, \quad Q_{j,\,i} = 2jL_i + 4j^2 M_i,$$
$$P'_{j,\,i} = (2j - c)\,L'_i + (2j - c)^2 M_i, \quad Q'_{j,\,i} = 2jL'_i + 4j^2 M_i.$$

These forms are given *in extenso* because the same transformations have been used to obtain the whole set of coefficients $A_{p,\,i}$ calculated. The quantities L_i, M_i, L'_i, M'_i are calculated once for all for each equation; it is then easy to obtain each coefficient $P_{j,\,i}$, etc. Their values are verified by the known results $P_{i,\,i} = P'_{i,\,i} = -1$. The coefficients once calculated, the work proceeds in the usual way by successive approximation, ε_i and ε'_i being found as linear functions of the unknowns ε_0 and ε'_0.

The coefficients a_i are, as has been mentioned, the same as Dr. Hill's $\mathrm{a}_i/\mathrm{a}_0$, and the numerical values of them are taken from his paper (Vol. I, p. 247).

It is useful to have an equation of verification at each step. Either of equations (6) or the second of equations (2) answer this purpose. One of equations (6) with suffix i enables us to verify ε_i, ε'_{-i}, ε_{-i} and ε'_i at once; and since the result of substituting the values obtained ought to render it an identity, ε_0 can be put equal to ε'_0 for this purpose. In this way the following table of results has been verified:

	ε_{-1}		ε'_1
1st approx.	$+.01998\ 37455\,\varepsilon_0 + .20559\ 07151\,\varepsilon'_0$		$-.01054\ 25540\,\varepsilon_0 - .07777\ 75955\,\varepsilon'_0$
2d "	$+.00001\ 51270\,\varepsilon_0 + .00008\ 82876\,\varepsilon'_0$		$-.00000\ 42515\,\varepsilon_0 - .00001\ 79511\,\varepsilon'_0$
3d "	$+.00000\ 00038\,\varepsilon_0 + .00000\ 00085\,\varepsilon'_0$		$-.00000\ 00003\,\varepsilon_0 + .00000\ 00036\,\varepsilon'_0$
$\varepsilon_{-1} =$	$+.01999\ 88763\,\varepsilon_0 + .20567\ 90112\,\varepsilon'_0$	$\varepsilon'_1 =$	$-.01054\ 68058\,\varepsilon_0 - .07779\ 55430\,\varepsilon'_0$

	ε_1		ε'_{-1}
1st approx.	$+.00308\ 02608\,\varepsilon_0 - .00092\ 87880\,\varepsilon'_0$		$-.00108\ 65604\,\varepsilon_0 - .00019\ 44186\,\varepsilon'_0$
2d "	$+.00000\ 00319\,\varepsilon_0 + .00000\ 07811\,\varepsilon'_0$		$-.00000\ 00355\,\varepsilon_0 - .00000\ 15807\,\varepsilon'_0$
3d "	$+.00000\ 00000\,\varepsilon_0 + .00000\ 00002\,\varepsilon'_0$		$-.00000\ 00001\,\varepsilon_0 - .00000\ 00006\,\varepsilon'_0$
$\varepsilon_1 =$	$+.00308\ 02927\,\varepsilon_0 - .00092\ 80067\,\varepsilon'_0$	$\varepsilon'_{-1} =$	$-.00108\ 65960\,\varepsilon_0 - .00019\ 59999\,\varepsilon'_0$

	ε_{-2}		ε'_2
1st approx.	$+.00001\ 15195\,\varepsilon_0 + .00007\ 34781\,\varepsilon'_0$		$-.00005\ 93615\,\varepsilon_0 - .00043\ 19502\,\varepsilon'_0$
2d "	$+.00000\ 00010\,\varepsilon_0 - .00000\ 00090\,\varepsilon'_0$		$-.00000\ 00261\,\varepsilon_0 - .00000\ 01280\,\varepsilon'_0$
$\varepsilon_{-2} =$	$+.00001\ 15205\,\varepsilon_0 + .00007\ 34691\,\varepsilon'_0$	$\varepsilon'_2 =$	$-.00005\ 93876\,\varepsilon_0 - .00043\ 20782\,\varepsilon'_0$

$$\begin{array}{rll}
 & \qquad\qquad \varepsilon_2 & \qquad\qquad \varepsilon'_{-2} \\
1^{\text{st}}\text{ approx.} & +.00001\ 47373\,\varepsilon_0 - .00000\ 85454\,\varepsilon'_0 & \quad +.00000\ 01043\,\varepsilon_0 - .00000\ 08608\,\varepsilon'_0 \\
2^{\text{d}}\quad\text{``} & \underline{+.00000\ 00003\,\varepsilon_0 + .00000\ 00076\,\varepsilon'_0} & \quad \underline{+.00000\ 00000\,\varepsilon_0 - .00000\ 00010\,\varepsilon'_0} \\
\varepsilon_2 = & +.00001\ 47376\,\varepsilon_0 - .00000\ 85378\,\varepsilon'_0 & \varepsilon'_{-2} = +.00000\ 01043\,\varepsilon_0 - .00000\ 08618\,\varepsilon'_0 \\
\varepsilon_{-3} = & -.00000\ 00193\,\varepsilon_0 - .00000\ 01734\,\varepsilon'_0 & \varepsilon'_3 = -.00000\ 04039\,\varepsilon_0 - .00000\ 29218\,\varepsilon'_0 \\
\varepsilon_3 = & +.00000\ 00843\,\varepsilon_0 - .00000\ 00708\,\varepsilon'_0 & \varepsilon'_{-3} = +.00000\ 00024\,\varepsilon_0 - .00000\ 00055\,\varepsilon'_0 \\
\varepsilon_{-4} = & -.00000\ 00001\,\varepsilon_0 - .00000\ 00012\,\varepsilon'_0 & \varepsilon'_4 = -.00000\ 00029\,\varepsilon_0 - .00000\ 00212\,\varepsilon'_0
\end{array}$$

A second approximation produces no effect in the last three coefficients. The advantage in point of accuracy is easily seen by the speed with which the coefficients approximate.

In equations (6) put $i = 0$; they become

$$\begin{aligned}
&\Sigma_j \left[\left\{c^2 + (2j + 1 + 2m)^2 + \tfrac{1}{2} m^2\right\} a_j + \tfrac{9}{2} m^2 a_{-j-1}\right] X_j + c\Sigma_j (2j + 1 + 2m)\, a_j Y_j = 0, \\
&\Sigma_j \left[c^2 a_j - 3m^2 a_{-j-1}\right] Y_j \qquad\qquad\qquad\qquad\qquad + c\Sigma_j (2j + 2 + 2m)\, a_j X_j = 0.
\end{aligned}$$

Remembering that

$$X_j = \varepsilon_j + \varepsilon'_j, \quad Y_j = \varepsilon_j - \varepsilon'_j,$$

these give on substitution of the values just obtained for ε_j, ε'_j,

$$\begin{aligned}
2.01291\ 56632\ 7\ X_0 + 1.24333\ 41788\ 2\ Y_0 &= 0, \\
2.31465\ 02085\ 6\ X_0 + 1.14989\ 92484\ 3\ Y_0 &= 0,
\end{aligned}$$

whence

$$\frac{Y_0}{X_0} = -\ 2.01291\ 56632\ 7,$$

$$\frac{Y_0}{X_0} = -\ 2.01291\ 56634\ 5.$$

The difference is probably due to accumulated errors. Taking the mean of these two values,

$$\frac{Y_0}{X_0} = -\ 2.01291\ 56633\ 6.$$

Hence with an error not exceeding one unit in the last place given,

$$\frac{X_0}{Y_0} = -\ .49679\ 18022.$$

We may therefore say, from the way that c is involved in the equations given above, *that the assumed value is certainly correct to the tenth place of decimals.*

As only one of the symbols ε_0, ε'_0, X_0, Y_0 can be arbitrary, we must decide which is the most convenient. Below it is shown that if e be the value of the eccentricity which Delaunay has used, Y_0 differs very little indeed from $2e$. In the case of our moon, Y_0 will therefore be just over .1, a convenient number to deal with. Hence Y_0 seems the best suited to be the arbitrary constant, and all the coefficients will be expressed in terms of it.

From the ratio X_0/Y_0 we find the ratios ε_0/Y_0, ε'_0/Y_0, and substituting in the table given above, we obtain as the final values of the coefficients the following:

$$\frac{\varepsilon_0}{Y_0} = +.25160\ 40989, \qquad \frac{\varepsilon'_0}{Y_0} = -.74839\ 59011,$$

$$\frac{\varepsilon_{-1}}{Y_0} = -.14889\ 75297, \qquad \frac{\varepsilon'_1}{Y_0} = +.05556\ 82459,$$

$$\frac{\varepsilon_{+1}}{Y_0} = +.00146\ 95307, \qquad \frac{\varepsilon'_{-1}}{Y_0} = -.00012\ 67065,$$

$$\frac{\varepsilon_{-2}}{Y_0} = -.00005\ 20854, \qquad \frac{\varepsilon'_2}{Y_0} = +.00030\ 84234,$$

$$\frac{\varepsilon_2}{Y_0} = +.00001\ 00977, \qquad \frac{\varepsilon'_{-2}}{Y_0} = +.00000\ 06713,$$

$$\frac{\varepsilon_{-3}}{Y_0} = +.00000\ 01250, \qquad \frac{\varepsilon'_3}{Y_0} = +.00000\ 20851,$$

$$\frac{\varepsilon_3}{Y_0} = +.00000\ 00742, \qquad \frac{\varepsilon'_{-3}}{Y_0} = +.00000\ 00048,$$

$$\frac{\varepsilon_{-4}}{Y_0} = +.00000\ 00009, \qquad \frac{\varepsilon'_4}{Y_0} = +.00000\ 00243.$$

Let $$D = (n - n')(t - t_1), \qquad l = c(n - n')(t - t_0),$$

and let v denote the excess of the true over the mean longitude.

The values just obtained give for the portion depending only on Y_0,

$$\begin{aligned} r\cos v = a_0 Y_0 [& -.49679\ 18022 \cos l \\ & -.09332\ 92838 \cos(2D - l) + .00134\ 28242 \cos(2D + l) \\ & +.00025\ 63380 \cos(4D - l) + .00001\ 07690 \cos(4D + l) \\ & +.00000\ 22101 \cos(6D - l) + .00000\ 00790 \cos(6D + l) \\ & +.00000\ 00252 \cos(8D - l) + .00000\ 00005 \cos(8D + l) \\ & +.00000\ 00003 \cos(10D - l)], \end{aligned}$$

$$
\begin{aligned}
r\cos v = a_0 Y_0 [&+ 1.00000\ 00000 \sin l \\
&+ .20446\ 57756 \sin(\ 2D - l) + .00159\ 62372 \sin(2D + l) \\
&+ .00036\ 05088 \sin(\ 4D - l) + .00000\ 94264 \sin(4D + l) \\
&+ .00000\ 19601 \sin(\ 6D - l) + .00000\ 00694 \sin(6D + l) \\
&+ .00000\ 00234 \sin(\ 8D - l) + .00000\ 00004 \sin(8D + l) \\
&+ .00000\ 00003 \sin(10D - l)].
\end{aligned}
$$

The coefficients of the terms with arguments $8D + l$ and $10D - l$ have been added by induction.

Dr. Hill suggested (Vol. I, p. 145) that a_0 should play the part usually assigned to a in the Lunar Theory. In the same way it seems best that Y_0 should be regarded as the arbitrary constant of this solution. The above then is complete so far as the first power of Y_0 is concerned.

It will be useful to compare these expressions with the corresponding ones obtained by Delaunay.* For this purpose the values found above must be transferred to polar coordinates. We have

$$\tan v = y'/x'.$$

In y' and x' are included the terms depending on m only. Denote by the operator δ the aggregate of the new terms added, depending on the first power of Y_0, then

$$\delta v = \frac{x'\delta y' - y'\delta x'}{x'^2 + y'^2}.$$

In this formula x', y' denote the values of $r\cos v$, $r\sin v$ when Y_0 is neglected, and $\delta x'$, $\delta y'$ the values when the terms depending only on Y_0 are taken into account. The value of δv is most easily found by giving to $2D$ the values 0, 30, 90, 150, 180 successively and calculating the coefficients of $\sin l$ and $\cos l$ in the several cases. In this way are obtained the series of coefficients in longitude. That of $\sin l$ comes out to be

$$.99972\ 87063\ Y_0.$$

Delaunay† fixes the coefficient of $\sin l$ as that which would be obtained in a purely elliptic motion. To the first power of the eccentricity this is $2e$. Hence

$$2e = .99972\ 87063\ Y_0,$$

and therefore

$$Y_0 = 2.00054\ 2735\quad e.$$

---

* Mémoires de l'Académie des Sciences, t. XXIX, pp. 818, 821, etc.

† Delaunay, Mém. cit., p. 798.

Using Delaunay's value of e, namely,

$$e = .05489\ 930,$$

we obtain the following series of coefficients in longitude:

$$+ 4607''.984 \sin(2D - l) + 35''.2200 \sin(4D - l)$$
$$+ 0''.2906 \sin(6D - l) + 0''.0027 \sin(8D - l)$$
$$+ 174''.8610 \sin(2D + l) + 1''.4460 \sin(4D + l)$$
$$+ 0''.0121 \sin(6D + l) + 0''.0001 \sin(8D + l).$$

With the assumed value of e, these are correct to the last place of decimals given. The coefficient of $\sin(8D + l)$ is added by induction. Delaunay's values for the corresponding parts are

$$+ 4607.771 \sin(2D - l) + 35''.1542 \sin(4D - l) + 0''.2174 \sin(6D - l)$$
$$+ 174''.8660 \sin(2D + l) + 1''.4094 \sin(4D + l) + 0''.0055 \sin(6D + l).$$

[*To be continued.*]

HAVERFORD COLLEGE, *December*, 1892.